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Nonlinear diffusion and geometry of domain *

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS3]. Let Ω be a C^2 domain in $\mathbb{R}^N$ with $N \geq 2$, and let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0, \quad \text{and} \quad 0 < \delta_1 \leq \phi'(s) \leq \delta_2 \quad \text{for } s \in \mathbb{R}, \tag{1.1}$$

where δ_1, δ_2 are positive constants. Consider the unique bounded solution $u = u(x, t)$ of either the initial-boundary value problem:

$$\partial_t u = \Delta\phi(u) \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1.2}$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{1.3}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.4}$$

or the initial value problem:

$$\partial_t u = \Delta\phi(u) \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u = \chi_{\Omega^c} \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \tag{1.5}$$

where χ_{Ω^c} denotes the characteristic function of the set $\Omega^c = \mathbb{R}^N \setminus \Omega$. By the maximum principle, we know that

$$0 < u < 1 \text{ either in } \Omega \times (0, +\infty) \text{ or in } \mathbb{R}^N \times (0, +\infty). \tag{1.6}$$

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Here, we have

$$\int_0^1 \frac{\phi'(\xi)}{\xi} d\xi = +\infty, \tag{1.7}$$

which means that the equation $\partial_t u = \Delta\phi(u)$ has the property of *infinite* speed of propagation of disturbances from rest. Let $\Phi = \Phi(s)$ be the function defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi} d\xi \quad \text{for } s > 0. \tag{1.8}$$

Note that if $\phi(s) \equiv s$, then $\Phi(s) = \log s$. This is the case corresponding to the heat equation. Define the distance function $d = d(x)$ by

$$d(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega. \tag{1.9}$$

Then, in [MS2] we have a generalization of a result of Varadhan [Va] to the nonlinear diffusion equation.

Theorem 1.1 ([MS2, Theorem 1.1 and Theorem 4.1]) *Let u be the solution of either problem* (1.2)-(1.4) *or problem* (1.5). *Then*

$$\lim_{t\to 0^+} -4t\Phi(u(x,t)) = d(x)^2 \tag{1.10}$$

uniformly on every compact set in Ω.

This theorem gives us an interaction between nonlinear diffusion and geometry of domain, since the distance function $d(x)$ is deeply related to the geometry of Ω.

Remark. In [MS2], only the case where $\partial\Omega$ is bounded is treated. *Here, let us show that Theorem* 1.1 *holds also when $\partial\Omega$ is unbounded.*

Take any point $x_0 \in \Omega$. For each $\varepsilon > 0$, there exist a point $z \in \mathbb{R}^N \setminus \overline{\Omega}$ and $\delta > 0$ such that $|x_0 - z| < d(x_0) + \varepsilon$ and $B_\delta(z) \subset \mathbb{R}^N \setminus \overline{\Omega}$, where $B_\delta(z)$ denotes the open ball in $\mathbb{R}^N$ with radius δ and centered at z.

Consider problem (1.2)-(1.4) first. Let $u^\pm = u^\pm(x,t)$ be bounded solutions of the following initial-boundary value problems:

$$\partial_t u^+ = \Delta\phi(u^+) \quad \text{in } B_{d(x_0)}(x_0) \times (0, +\infty), \tag{1.11}$$

$$u^+ = 1 \quad \text{on } \partial B_{d(x_0)}(x_0) \times (0, +\infty), \tag{1.12}$$

$$u^+ = 0 \quad \text{on } B_{d(x_0)}(x_0) \times \{0\}, \tag{1.13}$$

and

$$\begin{aligned}
\partial_t u^- &= \Delta\phi(u^-) && \text{in } \left(\mathbb{R}^N \setminus \overline{B_\delta(z)}\right) \times (0,+\infty), && (1.14)\\
u^- &= 1 && \text{on } \partial B_\delta(z) \times (0,+\infty), && (1.15)\\
u^- &= 0 && \text{on } \left(\mathbb{R}^N \setminus \overline{B_\delta(z)}\right) \times \{0\}, && (1.16)
\end{aligned}$$

respectively. Then it follows from the comparison principle that

$$u^-(x_0,t) \le u(x_0,t) \le u^+(x_0,t) \ \text{ for every } \ t > 0, \tag{1.17}$$

which gives

$$-4t\Phi(u^-(x_0,t)) \ge -4t\Phi(u(x_0,t)) \ge -4t\Phi(u^+(x_0,t)) \ \text{ for every } \ t > 0.$$

By [MS2, Theorem 1.1], letting $t \to 0^+$ yields that

$$(d(x_0)+\varepsilon)^2 \ge \limsup_{t\to 0^+}(-4t\Phi(u(x_0,t)) \ge \liminf_{t\to 0^+}(-4t\Phi(u(x_0,t)) \ge d(x_0)^2.$$

This implies (1.10). By a scaling argument, for each $0 < \rho_0 \le \rho_1 < +\infty$, in every subset E of $\{x \in \Omega \ : \ \rho_0 \le d(x) \le \rho_1\}$ where $\delta > 0$ can be chosen independently of each point $x \in E$, the convergence in (1.10) is uniform.

It remains to consider problem (1.5). Let $u^\pm = u^\pm(x,t)$ be bounded solutions of the following initial value problems:

$$\partial_t u^+ = \Delta\phi(u^+) \ \text{ in } \ \mathbb{R}^N \times (0,+\infty) \quad \text{ and } \ u^+ = \chi_{B_{d(x_0)}(x_0)^c} \ \text{ on } \ \mathbb{R}^N \times \{0\}, \tag{1.18}$$

and

$$\partial_t u^- = \Delta\phi(u^-) \ \text{ in } \ \mathbb{R}^N \times (0,+\infty) \quad \text{ and } \ u^- = \chi_{\overline{B_\delta(z)}} \ \text{ on } \ \mathbb{R}^N \times \{0\}, \tag{1.19}$$

respectively. Then by the comparison principle we get (1.17). Thus, with the aid of [MS2, Theorem 4.1], this gives us the conclusion (1.10) similarly.

Let us state our main theorem which also gives us another interaction between nonlinear diffusion and geometry of domain.

Theorem 1.2 ([MS3]) *Let u be the solution of either problem* (1.2)-(1.4) *or problem* (1.5). *Let $x_0 \in \Omega$. Assume that $B_R(x_0) \subset \Omega$ and $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some*

$y_0 \in \partial\Omega$. Then we have

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t)\,dx = c(\phi,N)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y_0)\right)\right\}^{-\frac{1}{2}}, \tag{1.20}$$

where $\kappa_1(y_0),\dots,\kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at $y_0 \in \partial\Omega$ with respect to the interior normal direction to $\partial\Omega$, and $c(\phi,N)$ is a positive constant depending only on ϕ and N(Of course, $c(\phi,N)$ depends on the problems (1.2)-(1.4) *and* (1.5)*). When $\kappa_j(y_0)=\frac{1}{R}$ for some $j\in\{1,\cdots,N-1\}$, the formula* (1.20) *holds by setting the right-hand side to be $+\infty$.*

Remark. Notice that we have

$$\kappa_j(y_0) \le \frac{1}{R} \quad \text{for every } j \in \{1,\cdots,N-1\}.$$

When $\partial\Omega$ is bounded and ϕ satisfies either $\int_0^1 \frac{\phi'(\xi)}{\xi}d\xi < +\infty$ or $\phi(s)\equiv s$, Theorem 1.2 was proved in [MS1] for problem (1.2)-(1.4). The method of the proof of the present article will enable us to show the same results also when $\partial\Omega$ is unbounded. In [MS1], the supersolutions and subsolutions to problem (1.2)-(1.4) were constructed in $\Omega_\rho\times(0,\tau]$ for sufficiently small $\rho>0$ and $\tau>0$, where

$$\Omega_\rho = \{x\in\Omega\ :\ d(x)<\rho\}. \tag{1.21}$$

In those processes, the property of *finite* speed of propagation of disturbances from rest coming from $\int_0^1 \frac{\phi'(\xi)}{\xi}d\xi < +\infty$ plays a useful role, since on $\Gamma_\rho\times(0,\tau]$ the solution u equals zero, where we put

$$\Gamma_\rho = \{x\in\Omega\ :\ d(x)=\rho\}. \tag{1.22}$$

Therefore, the estimates on $\Gamma_\rho\times(0,\tau]$ were easy. In the case where $\phi(s)\equiv s$, both the linearity of the heat equation and the result of Varadhan [Va] were used in constructing the supersolutions and subsolutions. Here, by using Theorem 1.1, we construct the supersolutions and subsolutions.

2 Outline of the proof of Theorem 1.2

In this section we give an outline of the proof of Theorem 1.2. For the details, see [MS3]. We distinguish two cases:

$$\text{(I) } \partial\Omega \text{ is bounded;} \quad \text{(II) } \partial\Omega \text{ is unbounded.}$$

Let us show that case (I) implies case (II). We can find two C^2 domains, say Ω_1, Ω_2, having bounded boundaries such that Ω_1 and $\mathbb{R}^N \setminus \overline{\Omega_2}$ are bounded, $B_R(x_0) \subset \Omega_1 \subset \Omega \subset \Omega_2$, and there exists $\beta > 0$ satisfying

$$B_\beta(y_0) \cap \partial\Omega \subset \partial\Omega_1 \cap \partial\Omega_2 \text{ and } \overline{B_R(x_0)} \cap \left(\mathbb{R}^N \setminus \Omega_i\right) = \{y_0\} \text{ for } i = 1, 2. \quad (2.1)$$

Let $u_i = u_i(x,t)$ $(i = 1, 2)$ be the two bounded solutions of either problem (1.2)-(1.4) or problem (1.5) where Ω is replaced by Ω_1, Ω_2, respectively. Since $\Omega_1 \subset \Omega \subset \Omega_2$, it follows from the comparison principle that

$$u_2 \le u \text{ in } \Omega \times (0, +\infty) \text{ and } u \le u_1 \text{ in } \Omega_1 \times (0, +\infty).$$

Therefore, it follows that for every $t > 0$

$$t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_2(x,t)\, dx \le t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t)\, dx \le t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_1(x,t)\, dx,$$

which shows that case (I) implies case (II).

Thus it suffices to consider case (I). We distinguish two cases:

$$\text{(IBVP) problem (1.2)-(1.4) ; (IVP) problem (1.5).}$$

Let us consider case (IBVP) first. We quote a result from Atkinson and Peletier [AtP]. It was shown in [AtP] that, for every $c > 0$, there exists a unique C^2 solution $f_c = f_c(\xi)$ of the problem:

$$(\phi'(f_c)f_c')' + \frac{1}{2}\xi f_c' = 0 \text{ in } [0, +\infty), \quad (2.2)$$

$$f_c(0) = c, \quad f_c(\xi) \to 0 \text{ as } \xi \to +\infty, \quad (2.3)$$

$$f_c' < 0 \text{ in } [0, +\infty). \quad (2.4)$$

By writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in [0, +\infty)$, we have:

$$-v_c'(0) = \frac{1}{2}\int_0^\infty f_c(s)\, ds \;\text{ for }\; c > 0; \tag{2.5}$$
$$0 < f_{c_1} < f_{c_2} \;\text{ on }\; [0, +\infty) \;\text{ if }\; 0 < c_1 < c_2 < +\infty; \tag{2.6}$$
$$0 > v_{c_1}'(0) > v_{c_2}'(0) \;\text{ if }\; 0 < c_1 < c_2 < +\infty. \tag{2.7}$$

Furthermore, [AtP, Lemma 4, p. 383] tells us that, for every compact interval I contained in $(0, +\infty)$,

$$\frac{-4\Phi(f_c(\xi))}{\xi^2} \to 1 \;\text{ as }\; \xi \to +\infty \;\text{ uniformly for }\; c \in I. \tag{2.8}$$

Note that, if we put $w(s,t) = f_c\left(t^{-\frac{1}{2}}s\right)$ for $s > 0$ and $t > 0$, then w satisfies the one-dimensional problem:

$\partial_t w = \partial_s^2 \phi(w)$ in $(0, +\infty)^2$, $w = c$ on $\{0\} \times (0, +\infty)$, and $w = 0$ on $(0, +\infty) \times \{0\}$.

Let $0 < \varepsilon < \frac{1}{4}$. We can find a sufficiently small $0 < \eta_\varepsilon << \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \geq 0$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon}\left(\sqrt{1 \mp 2\eta_\varepsilon}\; \xi\right) \quad \text{if } \xi \geq \eta_\varepsilon; \tag{2.9}$$
$$f_\pm' < 0 \;\text{ in }\; [0, +\infty); \tag{2.10}$$
$$f_- < f_1 < f_+ \;\text{ in }\; [0, +\infty); \tag{2.11}$$
$$\left(\phi'(f_\pm)f_\pm'\right)' + \frac{1}{2}\xi f_\pm' = h_\pm(\xi) f_\pm' \;\text{ in }\; [0, +\infty), \tag{2.12}$$

where $h_\pm = h_\pm(\xi)$ is defined by

$$h_\pm(\xi) = \begin{cases} \pm\eta_\varepsilon \xi & \text{if } \xi \geq \eta_\varepsilon, \\ \pm\eta_\varepsilon^2 & \text{if } \xi \leq \eta_\varepsilon. \end{cases} \tag{2.13}$$

Here, in order to use the function $h_\pm$ also for case (IVP) later, we defined $h_\pm(\xi)$ for all $\xi \in \mathbb{R}$. Then we notice that

$$h_+ = -h_- \geq \eta_\varepsilon^2 \;\text{ on }\; \mathbb{R} \;\text{ and }\; f_\pm \to f_1 \;\text{ as }\; \varepsilon \to 0 \;\text{ uniformly on }\; [0, +\infty). \tag{2.14}$$

Set $\Psi = \Phi^{-1}$. Then it follow from (2.8) that there exists $\xi_\varepsilon > 1$ such that

$$\Psi\left(-\frac{\xi^2}{4}\left(1 - \frac{\eta_\varepsilon}{2}\right)\right) > f_c(\xi) > \Psi\left(-\frac{\xi^2}{4}\left(1 + \frac{\eta_\varepsilon}{2}\right)\right) \quad \text{for } \xi \geq \xi_\varepsilon \text{ and } c \in I_\varepsilon, \tag{2.15}$$

where we set $I_\varepsilon = [1-2\varepsilon, 1+2\varepsilon]$.

Since $\partial\Omega$ is of class C^2 and bounded, there exists $\rho_0 > 0$ such that the distance function d belongs to $C^2(\overline{\Omega_{\rho_0}})$. Set $\rho_1 = \max\{2R, \rho_0\}$. Theorem 1.1 yields that

$$-4t\Phi(u(x,t)) \to d(x)^2 \quad \text{as } t \to 0^+ \quad \text{uniformly on } \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}. \tag{2.16}$$

Then there exists $\tau_{1,\varepsilon} > 0$ such that for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$

$$\left|-4t\Phi(u(x,t)) - d(x)^2\right| < \frac{1}{2}\eta_\varepsilon \rho_0^2 \le \frac{1}{2}\eta_\varepsilon d(x)^2.$$

Thus it follows that for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$

$$\Psi\left(-\frac{\left(1-\frac{1}{2}\eta_\varepsilon\right)}{4}\frac{d(x)^2}{t}\right) > u(x,t) > \Psi\left(-\frac{\left(1+\frac{1}{2}\eta_\varepsilon\right)}{4}\frac{d(x)^2}{t}\right). \tag{2.17}$$

From (2.15), we have

$$f_+(\xi) = f_{1+\varepsilon}(\sqrt{1-2\eta_\varepsilon}\ \xi) > \Psi\left(-\frac{\xi^2}{4}\left(1-\frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \ge \frac{\xi_\varepsilon}{\sqrt{1-2\eta_\varepsilon}}; \tag{2.18}$$

$$f_-(\xi) = f_{1-\varepsilon}(\sqrt{1+2\eta_\varepsilon}\ \xi) < \Psi\left(-\frac{\xi^2}{4}\left(1+\frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \ge \frac{\xi_\varepsilon}{\sqrt{1+2\eta_\varepsilon}}. \tag{2.19}$$

Define the two functions $w_\pm = w_\pm(x,t)$ by

$$w_\pm(x,t) = f_\pm\left(t^{-\frac{1}{2}}d(x)\right) \quad \text{for } (x,t) \in \Omega \times (0,+\infty). \tag{2.20}$$

Hence it follows from (2.17), (2.18) and (2.19) that there exists $\tau_{2,\varepsilon} \in (0, \tau_{1,\varepsilon}]$ satisfying

$$w_- < u < w_+ \quad \text{in } \left(\overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}\right) \times (0, \tau_{2,\varepsilon}]. \tag{2.21}$$

Since $d \in C^2(\overline{\Omega_{\rho_0}})$ and $|\nabla d| = 1$ in $\overline{\Omega_{\rho_0}}$, we have

$$\partial_t w_\pm - \Delta\phi(w_\pm) = -f_\pm' t^{-1}\left\{h_\pm + \sqrt{t}\ \phi'(f_\pm)\Delta d\right\} \quad \text{in } \overline{\Omega_{\rho_0}} \times (0,+\infty). \tag{2.22}$$

Therefore, it follows from the former formula of (2.14) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying

$$\partial_t w_- - \Delta\phi(w_-) < 0 < \partial_t w_+ - \Delta\phi(w_+) \quad \text{in } \Omega_{\rho_0} \times (0, \tau_{3,\varepsilon}]. \tag{2.23}$$

Observe that

$$w_- = u = w_+ = 0 \ \text{ in } \ \Omega_{\rho_0} \times \{0\}, \tag{2.24}$$

$$w_- = f_-(0) < 1 = f_1(0) = u < f_+(0) = w_+ \ \text{ on } \ \partial\Omega \times (0, \tau_{3,\varepsilon}], \tag{2.25}$$

$$w_- < u < w_+ \ \text{ on } \ \Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}]. \tag{2.26}$$

Note that (2.26) comes from (2.21). Thus it follows from the comparison principle and (2.21) that

$$w_- \le u \le w_+ \ \text{ in } \ \overline{\Omega_{\rho_1}} \times (0, \tau_{3,\varepsilon}]. \tag{2.27}$$

Here we quote a geometric lemma from [MS1] adjusted to our situation:

Lemma 2.1 ([MS1, Lemma 2.1, p. 376]) *Suppose that $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$. Then we have:*

$$\lim_{s\to 0+} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_s \cap B_R(x_0)) = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}, \tag{2.28}$$

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure, and ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

First of all, let us consider the case where $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$. It follows from (2.27) that

$$\int_{B_R(x_0)} w_- \, dx \le \int_{B_R(x_0)} u \, dx \le \int_{B_R(x_0)} w_+ \, dx \ \text{ for every } t \in (0, \tau_{3,\varepsilon}]. \tag{2.29}$$

Since with the aid of the co-area formula we have

$$\int_{B_R(x_0)} w_\pm \, dx = t^{\frac{N+1}{4}} \int_0^{2Rt^{-\frac{1}{2}}} f_\pm(\xi) \xi^{\frac{N-1}{2}} \left(t^{\frac{1}{2}} \xi \right)^{-\frac{N-1}{2}} \mathcal{H}^{N-1} \left(\Gamma_{t^{\frac{1}{2}}\xi} \cap B_R(x_0) \right) d\xi,$$

by using Lebesgue's dominated convergence theorem and Lemma 2.1 we get

$$\lim_{t\to 0+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} w_\pm \, dx = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}} \int_0^\infty f_\pm(\xi) \xi^{\frac{N-1}{2}} d\xi.$$

Therefore, since $\varepsilon > 0$ is arbitrarily small, the latter formula of (2.14) yields (1.20), where we set

$$c(\phi, N) = 2^{\frac{N-1}{2}} \omega_{N-1} \int_0^\infty f_1(\xi) \xi^{\frac{N-1}{2}} d\xi.$$

It remains to consider the case where $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \cdots, N-1\}$. Choose a sequence of balls $\{B_{R_k}(x_k)\}_{k=1}^{\infty}$ satisfying:

$R_k < R$, $y_0 \in \partial B_{R_k}(x_k)$ and $B_{R_k}(x_k) \subset B_R(x_0)$ for every $k \geq 1$, and $\lim_{k\to\infty} R_k = R$.

Since $\kappa_j(y_0) \leq \frac{1}{R} < \frac{1}{R_k}$ for every $j = 1, \ldots, N-1$ and every $k \geq 1$, we can apply the previous case to each $B_{R_k}(x_k)$ to see that for every $k \geq 1$

$$\begin{aligned}
\liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t)\, dx &\geq \liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_{R_k}(x_k)} u(x,t)\, dx \\
&= c(\phi, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R_k} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}.
\end{aligned}$$

Hence, letting $k \to \infty$ yields that

$$\liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t)\, dx = +\infty,$$

which completes the proof for problem (1.2)-(1.4).

Let us consider case (IVP) and let $u = u(x,t)$ be the solution of problem (1.5). We replace problem (2.2)-(2.4) by the following problem for every $c > 0$:

$$(\phi'(f_c)f_c')' + \frac{1}{2}\xi f_c' = 0 \text{ in } \mathbb{R}, \tag{2.30}$$

$$f_c(\xi) \to c \text{ as } \xi \to -\infty, \quad f_c(\xi) \to 0 \text{ as } \xi \to +\infty, \tag{2.31}$$

$$f_c' < 0 \text{ in } \mathbb{R}. \tag{2.32}$$

By writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in \mathbb{R}$, we have:

$$-v_c'(0) = \frac{1}{2}\int_0^{\infty} f_c(s)\, ds \text{ for } c > 0; \tag{2.33}$$

$$0 < f_{c_1} < f_{c_2} \text{ on } \mathbb{R} \text{ if } 0 < c_1 < c_2 < +\infty; \tag{2.34}$$

$$0 > v_{c_1}'(0) > v_{c_2}'(0) \text{ if } 0 < c_1 < c_2 < +\infty. \tag{2.35}$$

Then [AtP, Lemma 4, p. 383] tells us that (2.8) also holds for the solution f_c of this problem. Note that if we put $w(s,t) = f_c\left(t^{-\frac{1}{2}}s\right)$ for $s \in \mathbb{R}$ and $t > 0$, then w satisfies the one-dimensional initial value problem:

$$\partial_t w = \partial_s^2 \phi(w) \text{ in } \mathbb{R} \times (0, +\infty) \text{ and } w = c\chi_{(-\infty,0]} \text{ on } \mathbb{R} \times \{0\}.$$

Let $0 < \varepsilon < \frac{1}{4}$. We can find a sufficiently small $0 < \eta_\varepsilon << \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \in \mathbb{R}$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon}\left(\sqrt{1 \mp 2\eta_\varepsilon}\ \xi\right) \quad \text{if } \xi \geq \eta_\varepsilon; \tag{2.36}$$

$$f'_\pm < 0 \quad \text{in } \mathbb{R}; \tag{2.37}$$

$$f_-(-\infty) < 1 = f_1(-\infty) < f_+(-\infty) \quad \text{and} \quad f_- < f_1 < f_+ \quad \text{in } \mathbb{R}; \tag{2.38}$$

$$(\phi'(f_\pm)f'_\pm)' + \frac{1}{2}\xi f'_\pm = h_\pm(\xi)f'_\pm \quad \text{in } \mathbb{R}. \tag{2.39}$$

Then we also have (2.14).

Moreover, it follows from (2.8) that there exists $\xi_\varepsilon > 1$ satisfying (2.15). Proceeding similarly yields (2.16), (2.17), (2.18) and (2.19). Let us consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \ \ \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases} \tag{2.40}$$

Since $\partial\Omega$ is of class C^2 and bounded, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2-smooth on a compact neighborhood $\mathcal{N}$ of the boundary $\partial\Omega$ given by

$$\mathcal{N} = \{x \in \mathbb{R}^N : -\rho_0 \leq d^*(x) \leq \rho_0\}. \tag{2.41}$$

For simplicity we have used the same $\rho_0 > 0$ as in (2.16). Define $w_\pm = w_\pm(x, t)$ by

$$w_\pm(x, t) = f_\pm\left(t^{-\frac{1}{2}}d^*(x)\right) \quad \text{for} \quad (x, t) \in \mathbb{R}^N \times (0, +\infty). \tag{2.42}$$

Then we also have (2.21). Since $d^* \in C^2(\mathcal{N})$ and $|\nabla d^*| = 1$ in $\mathcal{N}$, we have

$$\partial_t w_\pm - \Delta\phi(w_\pm) = -f'_\pm t^{-1}\left\{h_\pm + \sqrt{t}\ \phi'(f_\pm)\Delta d^*\right\} \quad \text{in } \mathcal{N} \times (0, +\infty). \tag{2.43}$$

Therefore, it follows from the former formula of (2.14) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying:

$$\partial_t w_- - \Delta\phi(w_-) < 0 < \partial_t w_+ - \Delta\phi(w_+) \quad \text{in } \mathcal{N} \times (0, \tau_{3,\varepsilon}], \tag{2.44}$$

$$w_- \leq u \leq w_+ \quad \text{in } \mathcal{N} \times \{0\}, \tag{2.45}$$

$$w_- < u < w_+ \quad \text{on } \partial\mathcal{N} \times (0, \tau_{3,\varepsilon}]. \tag{2.46}$$

Note that in (2.46) the inequality on $\Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}]$ comes from (2.21) and the inequality on $(\partial\mathcal{N} \setminus \Gamma_{\rho_0}) \times (0, \tau_{3,\varepsilon}]$ comes from the former formula of (2.38). Thus it follows from the comparison principle and (2.21) that

$$w_- \le u \le w_+ \ \text{ in } \overline{\mathcal{N} \cup \Omega_{\rho_1}} \times (0, \tau_{3,\varepsilon}]. \tag{2.47}$$

Then, with the aid of (2.47) the rest of the proof runs similarly.

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References

[AtP] F. V. Atkinson and L. A. Peletier, Similarity solutions of the nonlinear diffusion equation, Arch. Rational Mech. Anal. 54 (1974), 373–392.

[GT] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, (Second Edition.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[MS1] R. Magnanini and S. Sakaguchi, Interaction between degenerate diffusion and shape of domain, Proceedings Royal Soc. Edinburgh, Section A, 137 (2007), 373–388.

[MS2] R. Magnanini and S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. Henri Poincaré - (C) Anal. Non Linéaire, to appear.

[MS3] R. Magnanini and S. Sakaguchi, Interaction between nonlinear diffusion and geometry of domain, in preparation.

[Va] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. 20 (1967), 431–455.